

## 6 General properties of an autonomous system of two first order ODE

Here I embark on studying the autonomous system of two first order differential equations of the form

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2),\end{aligned}\tag{1}$$

where  $f_1, f_2 \in \mathcal{C}^{(1)}(U; \mathbf{R})$ ,  $U \subseteq \mathbf{R}^2$ ,  $x_1, x_2$  are unknown functions, and  $t$  is the independent variable which usually denotes time. System (1) can be conveniently written in the vector form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t) \in U \subseteq \mathbf{R}^2, \quad \mathbf{f}: U \longrightarrow \mathbf{R}^2,\tag{2}$$

where  $\mathbf{x}(t) = (x_1(t), x_2(t))^\top$ ,  $\top$  denotes transposition (all the vectors are assumed to be column-vectors),  $\mathbf{f} = (f_1, f_2)^\top$  is a vector-function of two variables, which maps a subset  $U$  of  $\mathbf{R}^2$  to  $\mathbf{R}^2$ , and the bold font usually denotes vectors (please be aware that in some books vectors are written in the same font as scalar variables, therefore it is the reader's task to figure out the dimensions of the object).

I will assume throughout the lectures that for system (1) (or, equivalently, for system (2)) the theorem of uniqueness and existence of solutions is satisfied. To be precise,

**Theorem 1.** *Consider problem (2) together with the initial condition*

$$\mathbf{x}(t_0) = \mathbf{x}_0 \in U \subseteq \mathbf{R}^2\tag{3}$$

*and assume that  $\mathbf{f} \in \mathcal{C}^{(1)}(U; \mathbf{R}^2)$ . Then there exists an  $\epsilon > 0$  such that solution to (2)–(3) exists and unique for  $t \in (t_0 - \epsilon, t_0 + \epsilon)$ .*

It is important to note that the theorem is local, it guarantees the existence and uniqueness of the solutions only on a small interval of  $t$ . According to the general theory I can usually extend this unique solution to some larger interval  $(t_0 - T^-, t_0 + T^+)$ , but, as simple examples show (e.g.,  $\dot{x} = x^2$ ),  $T^\pm$  do not have to be infinite. In other words, solutions can *blow up* in finite time. In the rest of these lectures I will safely ignore this fact by tacitly assuming that  $T^\pm = \pm\infty$ , and this always will be the case for all the models I will study. Therefore, I assume that problem (2)–(3) has a unique solution, which I denote

$$\mathbf{x}(t; \mathbf{x}_0),$$

which is defined for all  $t \in (-\infty, \infty)$ .

There are a lot of notions pertaining to (2) that I discussed already in the context of the first order ODE. The crucial distinctions will be clear a little later.

**Definition 2.** *The set  $U \subseteq \mathbf{R}^2$  in which the solutions to (2) are defined is called the phase space or the state space of system (2).*

By the definition of the solutions to (2)–(3) we have that they represent a curve in the phase space parameterized by the time  $t$ . These curves called *orbits* or *trajectories* of (2).

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**Definition 3.** The set

$$\gamma(\mathbf{x}_0) = \{\mathbf{x}(t; \mathbf{x}_0) : t \in \mathbf{R}\}$$

is called an orbit of (2) starting at the point  $\mathbf{x}_0$ . Positive and negative semi-orbits are defined as

$$\gamma^+(\mathbf{x}_0) = \{\mathbf{x}(t; \mathbf{x}_0) : t \in [t_0, \infty)\}$$

and

$$\gamma^-(\mathbf{x}_0) = \{\mathbf{x}(t; \mathbf{x}_0) : t \in (-\infty, t_0]\}$$

respectively.

Some of the orbits are quite special. For example,

**Definition 4.** A point  $\hat{\mathbf{x}}$  such that

$$\gamma(\hat{\mathbf{x}}) = \{\hat{\mathbf{x}}\}$$

is called an equilibrium point, or stationary point, or fixed point, or rest point of system (2).

It should be obvious that

**Proposition 5.** A point  $\hat{\mathbf{x}}$  is an equilibrium if and only if

$$\mathbf{f}(\hat{\mathbf{x}}) = 0.$$

A very important property of the autonomous system (2) is presented in the following

**Proposition 6.** If  $\phi(t) \in U \subseteq \mathbf{R}^2$  is a solution to (2), then  $\phi(t - t_0)$  is also a solution for any constant  $t_0$ .

*Proof.* I am given that

$$\frac{d}{dt}\phi(t) = \mathbf{f}(\phi(t)),$$

and I need to show that

$$\frac{d}{dt}\phi(t - t_0) = \mathbf{f}(\phi(t - t_0))$$

is also true. Let me make the change of the independent variable in the last expression:

$$\tau = t - t_0.$$

I have

$$\frac{d}{dt}\phi(\tau) = \frac{d}{d\tau}\phi(\tau) \frac{d\tau}{dt} = \mathbf{f}(\phi(\tau)),$$

or, since  $\frac{d\tau}{dt} = 1$ ,

$$\frac{d}{d\tau}\phi(\tau) = \mathbf{f}(\phi(\tau)),$$

which is exactly where I started, only with  $\tau$  instead of  $t$ . This means that  $\phi(\tau)$  is a solution, and hence, by returning to  $\tau = t - t_0$ , that  $\phi(t - t_0)$  is also a solution. ■

The orbits should not be confused with the *integral curves*: the graphs of the solutions in the space  $\mathbf{R} \times \mathbf{R}^2$  given parametrically as  $(t, x_1(t), x_2(t))$ . Here is a simple illustration: consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1.$$

I can actually find the solutions of this system in explicit form because the system is linear. Assume that I consider two different initial conditions:  $\mathbf{x}_1$  and  $\mathbf{x}_2$  (here is a very common source of confusion: I use subscripts to denote both the elements of the vectors, i.e.,  $\mathbf{x} = (x_1, x_2)$ , and distinguish two vectors, i.e.,  $\mathbf{x}_1 = (x_1^1, x_2^1)$  and  $\mathbf{x}_2 = (x_1^2, x_2^2)$ . Try not to confuse them.) Hence I have two solutions of the system:

$$\mathbf{x}_1(t) = \mathbf{x}_1(t; \mathbf{x}_1),$$

and

$$\mathbf{x}_2(t) = \mathbf{x}_2(t; \mathbf{x}_2).$$

If I plot these solutions in 3D space  $(t, x_1, x_2)$  I have the integral curves (see the figure, the red curves are the integral curves, and the red dots denote the initial conditions), if I plot them on the plane  $(x_1, x_2)$  then I have the orbits (the blue curves in the figure), which are the curves together with the directions specified by  $t$  (I do not put directions on the integral curves because there is a natural direction of the time increase). There is one equilibrium  $\hat{\mathbf{x}} = (0, 0)$ , which is shown by a blue dot, together with the corresponding integral curve, which is parallel to  $t$ -axis. Therefore, I can conclude that *the orbits are the projections of the integral curves on the phase space*.

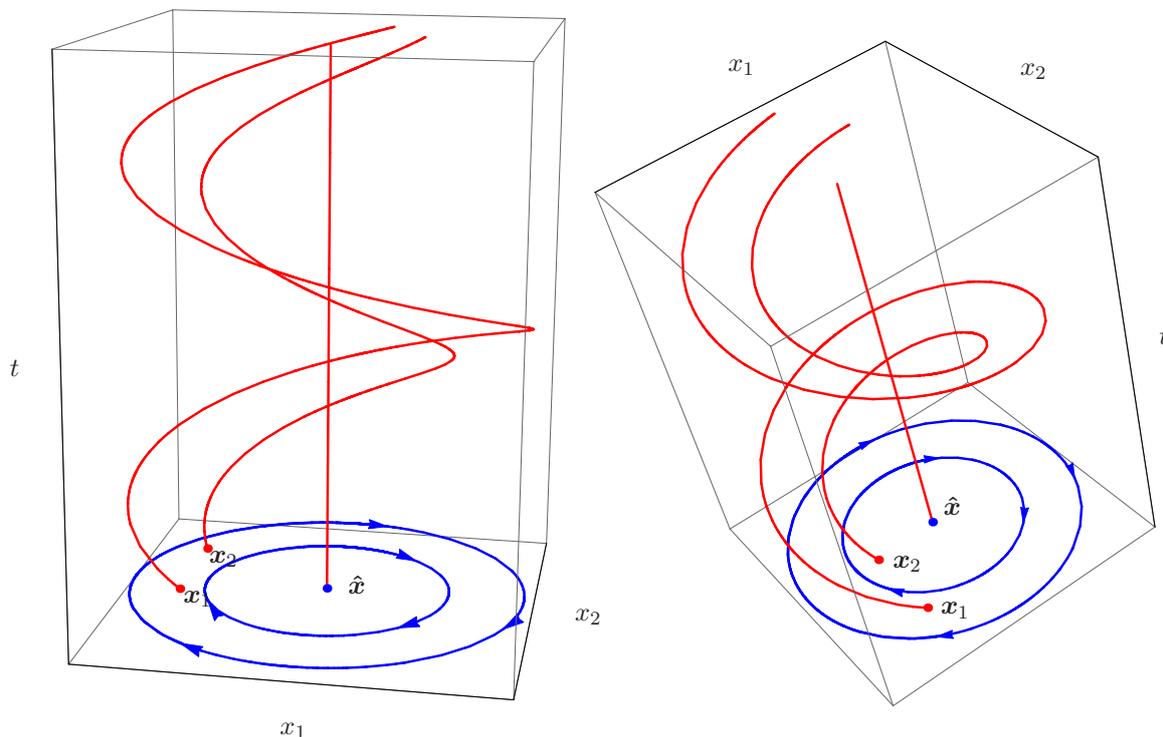


Figure 1: Integral (red) and phase curves (orbits, blue) of the system  $\dot{x}_1 = x_2, \dot{x}_2 = -x_1$ . See text for details

You can see in the figure that the orbits do not intersect. This is not an obvious observation in general, but it is true due to the fact that the system is autonomous.

**Proposition 7.** *Two orbits of (2) either do not have common points or coincide.*

*Proof.* Assume that  $\mathbf{x}_0 \in U \subseteq \mathbf{R}^2$  is a point that belongs to two orbits. This implies that there are two solutions  $\phi$  and  $\psi$  and  $t_1$  and  $t_2$  such that

$$\phi(t_1) = \psi(t_2).$$

Consider

$$\chi(t) = \phi(t + (t_1 - t_2)).$$

This function, due to Proposition 6, is also a solution. Moreover, the orbit corresponding to  $\chi$  coincides with the the orbit corresponding to  $\phi$  because of its definition (we just use a different parametrization on the orbit, shifted by the value  $t_1 - t_2$ ). On the other hand,

$$\chi(t_2) = \phi(t_1) = \psi(t_2),$$

which yields, due to the theorem of existence and uniqueness of the solutions that  $\chi$  and  $\psi$  coincide. Therefore the orbits defined by  $\phi$  and  $\psi$  coincide. ■

To repeat myself, the last proposition is not true for non-autonomous systems. This is why, while studying autonomous systems, I can restrict my attention to the phase space and the orbits in this space.

Now consider the group property of the solutions to (2).

**Proposition 8.** *If  $\mathbf{x}(t; \mathbf{x}_0)$  is a solution to (2), then*

$$\mathbf{x}(t_2 + t_1; \mathbf{x}_0) = \mathbf{x}(t_2; \mathbf{x}(t_1; \mathbf{x}_0)) = \mathbf{x}(t_1; \mathbf{x}(t_2; \mathbf{x}_0))$$

for any  $t_1, t_2 \in \mathbf{R}$ .

*Proof.* Consider two solutions to (2):

$$\phi_1(t) = \mathbf{x}(t; \mathbf{x}(t_1; \mathbf{x}_0)), \quad \phi_2(t) = \mathbf{x}(t + t_1; \mathbf{x}_0).$$

By the definition of solution:

$$\mathbf{x}(0; \mathbf{x}_0) = \mathbf{x}_0.$$

I have that

$$\phi_1(0) = \phi_2(0),$$

which, by theorem of uniqueness and existence implies that they coincide. If I plug  $t_2$  instead of  $t$  in these function, I obtain first of the required equalities. The second one is proved in a similar way. ■

The last proposition in particular shows that

$$\mathbf{x}(-t; \mathbf{x}(t; \mathbf{x}_0)) = \mathbf{x}_0.$$

In the language of algebra this means that the solution to (2), which is often called *the flow of the system*, forms a group acting on the phase (or state) space  $U$ . I will come back to this interpretation in due course.

Another useful viewpoint at the system (2) is to recognize that the right hand side actually defines a *vector field* at any point  $\mathbf{x} \in U$ , i.e., for each point  $\mathbf{x} \in U$  there is a vector  $(f_1(\mathbf{x}), f_2(\mathbf{x}))^\top$ , which is tangent to the orbit at this particular point and points in the direction of the time increase. I illustrate this concept with the same simple system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1.$$

Note that the only point at which the vector field is not defined is the equilibrium  $\hat{\mathbf{x}} = (0, 0)$ .

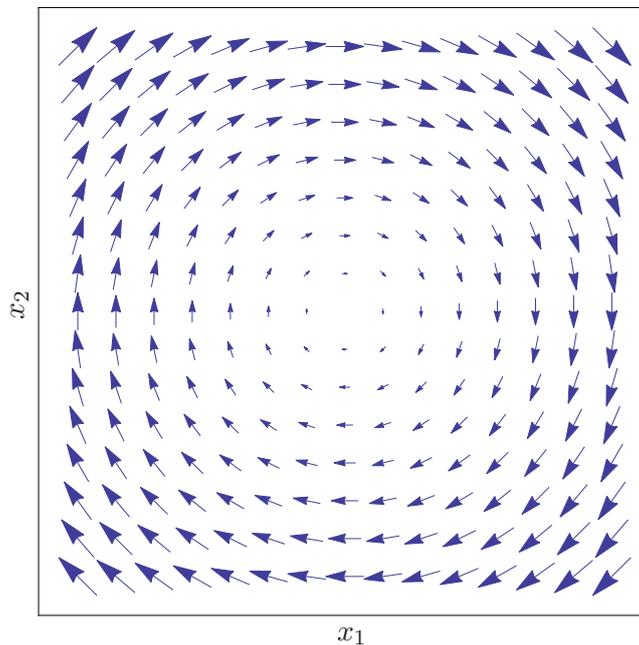


Figure 2: The vector field of the system  $\dot{x}_1 = x_2$ ,  $\dot{x}_2 = -x_1$  (cf. the previous figure)

Using the notion of the vector field I obtain almost immediate

**Proposition 9.** *Any orbit of (2) different from an equilibrium point is a smooth curve.*

And here by “smooth curve” I mean a curve that has a tangent vector at any point. Moreover,

**Proposition 10.** *Any orbit of (2) belongs to one of three types: a smooth curve without self-intersections, a closed smooth curve (cycle), or a point. The solution corresponding to the cycle is a periodic function of  $t$ .*

*Proof.* If the orbit is not a point then, according to the previous proposition, it is a smooth curve. The smooth curve is either closed or not. If it is closed and because of the fact that at any point of this curve there is non-zero constant with respect to  $t$  vector field, then there is a constant  $T > 0$  such that the solution corresponding to the closed orbit satisfies  $\mathbf{x}(t + T; \mathbf{x}_0) = \mathbf{x}(t; \mathbf{x}(T; \mathbf{x}_0)) = \mathbf{x}(t; \mathbf{x}_0)$ , i.e., a periodic function of the period  $T$ . ■

It is impossible to draw all the orbits in the phase space. However, I can get a general impression of the orbits' behavior by looking at several key orbits, such as equilibria, cycles, and orbits connecting equilibria.

**Definition 11.** *Partitioning the phase space into orbits is called the phase portrait.*

We already draw a number of phase portraits in the case of one first order ODE. Then it was just a partitioning of the  $x$ -axis into orbits. Here I have the plane as our phase space, therefore we have much more possibilities for the mutual positioning of the orbits (and therefore this problem is more complex than the one-dimensional case). You should look at the phase portrait of the Lotka–Volterra system to make sure you understand the meanings of the many concepts introduced in this lecture.

And final remark: I was talking about systems of two first order differential equations. However, nowhere in the proof I used the fact that the phase space is two dimensional. All the statements in this lecture hold for any dimension with an obvious caveat that already in three dimensions the phase portraits are quite difficult to present (although there will be some example), and in dimension four and higher this nice geometric interpretation is of almost no use.